Contest of Teams

XIV. Kavics Cup

13. 03. 2018.

Important rules

- Each answer should be an integer number between 0000 and 9999.
- If the result is greater than 9999, the answer is the last four digits of the number.
- If the result is negative, or the problem does not have a solution, or the problem has multiple solutions, write this: ????.
- These approximate values can be useful for the calculations:

$$\sqrt{2} \approx 1.4142 \quad \sqrt{3} \approx 1.7321 \quad \sqrt{5} \approx 2.2361 \quad \sqrt{7} \approx 2.6458 \quad \pi \approx 3.1416$$

Time limits

- The Jolly problem can be selected in the first 15 minutes.
- After the first 30 minutes questions regarding the text cannot be raised.
  Only the captains of the teams can ask questions directed at the jury.
- The length of the competition is 90 minutes.

Problem 1. Let n be the number of triangles such that each side has integral length and the longest side has length 11. (We do not count degenerate triangles.)
The answer is the least common multiple of n and 14. **(20 points)**

Problem 2. Captain Immortal has three immortal grandchildren, their age is p, q and r, which are all different primes, moreover $p^2 + q^2 + r^2$ is also prime. The youngest grandchild is p years old. All the ages of the mentioned 4 persons can be arbitrarily large.
The answer is $2018 - p \cdot 14$. **(20 points)**

Problem 3. The function $f(x) = |1 - 2x|$ is defined on the interval $[0; 1]$. The equation below has n solutions.

$$f(f(f(x))) = x/2$$

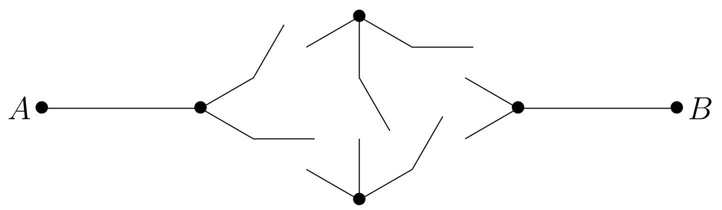
The answer is $14n$. **(20 points)**

Problem 4. P is a point on the plane of the square $ABCD$ such that each of PAB, PBC, PCD, PDA is an isosceles triangle (none of them is degenerate). Let n be the number of positions for such a point P.
The answer is $\left[10000\left\{\frac{n}{14}\right\}\right]$. **(25 points)**

Problem 5. Consider the equation $2x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + x_{10} = 3$. Let n be the number of non-negative integral solutions.
The answer is $n + 14$. **(25 points)**

Problem 6.

The figure below shows a detail of a current circuit. All switches are open or closed with probability $\frac{1}{2}$, their position is independent from each other. Let $\frac{m}{n}$ be the the probability $(gcd(m;n) = 1)$), that the current goes from A to B.
The answer is $n + m + 2018$.



(25 points)

Problem 7. Determine the two smallest positive integers $x < y$ whith the following property:

Multiply the number with 13 and write it down in base 7. The last two digits of it should be 43.
The answer is $\overline{xy}$.

(30 points)

Problem 8.

Let $ABCD$ be a quadrilateral. Denote the centroids of BCD, ACD, ABD and ABC by A_1, B_1, C_1 and D_1, respectively. Let m/n be the ratio of the area of $A_1B_1C_1D_1$ and $ABCD$ such that $gcd(m;n) = 1$.
The answer is $14n + m$. **(30 points)**

Problem 9. For any two real numbers x and y, define $x \sim y = ax + by + cxy$, where a, b, c are constants. It is known that $1 \sim 2 = 3$ and $2 \sim 3 = 4$ and there is a non-zero real number d such that $x \sim d = x$ for any real number x.
The answer is $d \sim (-2018)$.

(30 points)

Problem 10.

Each of sixteen cities entered an A team and a B team in a soccer tournament. Any two teams were to play each other once, except for teams from the same city which would not play each other. At some point during the tournament, the A team of a certain city noticed that every other team had played a different number of games. Let n be the number of games which had been played by the B team of this city.
The answer is $n + 14$.

(35 points)

Problem 11. In a triangle ABC $\sin A = 3/5$ and $\cos B = 5/13$. The value of $\cos C = p/q$, expressed in the lowest terms.
The answer is $\overline{pq}$.

(35 points)

Problem 12. Let n be the number of blocks of consecutive terms in the sequence 1, 4, 8, 10, 16, 8, 21, 25, 30, 43 with sums divisible by 11.
The answer is $2018 - n$. **(35 points)**

Problem 13. Let n be the number of distinct solutions of $\cos(x/4) = \cos x$ in the interval $(0; 24\pi)$.
The answer is $n + 14^2$. **(35 points)**

Problem 14.

Consider a regular octahedron, the length of each edge is 3. Chop off a square pyramid with each edge 1 at all of the vertices. This remaining polyhedron has k edges, write the numbers 1,2,...,k on the edges. Let n be the number of pairs $(i; j)$ $(1 \le i < j \le k)$ such that the lines of the edges with i and j are skew line.
The answer is $n + 1144$.

(40 points)

Problem 15. A country consists of N islands $A_1, A_2, \ldots, A_N$. The Ministry of Transport decides to build some bridges so that it will be possible to travel by car from each of the N islands to any other island by using one or more bridges. For technical reasons, the only possible bridges that might be built are from A_i to A_{i+1} for $i = 1, 2 \ldots, N-1$ and from A_i to A_N where $i < N$.

A plan to build a collection of bridges is said to be *good* if it fulfils the given conditions, but if any one bridge is omitted, then it does not. Suppose that there are a_N good plans. Notice that $a_1 = 1$ (the only good plan is to build no bridge), and $a_2 = 1$ (build a bridge).

The answer is $a_6 + 14$. **(40 points)**

Problem 16. 8 people are sitting in a row, they are from 4 countries, two people from each of the countries. Let n be the number of such permutations of the 8 people which satisfies that any two people sitting next to each other are from different countries.

The answer is $n - 10000$.

(45 points)

Problem 17. Consider the following two sets: $A = \{1, 2, 3, 4, 5\}$ and $B = \{1, 2, 3\}$. The domain of function f is A and $f(x) \in A$ if $x \in A$. Let n be the number of those functions for which the range of $f(f(x))$ is B.

The answer is $n + 1414$.

(45 points)

Problem 18. Let $ABCD$ be a 3×10 rectangle. We would like to tile it with 2×1 dominoes. Let n be the number of ways we can cover $ABCD$.

The answer is $14n$. **(45 points)**

Problem 19. All faces of a hexahedron and a regular octahedron are equilateral triangles of the same size. Expressed as a fraction in the lowest terms the ratio of the inradii of these two polyhedra is m/n.

The answer is $14mn + 14m + n$. **(50 points)**

Problem 20.

$$f(x) = \frac{4^x}{4^x + 2}, \quad n = \sum_{i=1}^{2018} f\left(\frac{i}{2019}\right).$$

The answer is $2018 + n$. **(50 points)**